A NEW METHOD FOR THE ABSOLUTE MEASUREMENT OF ELECTRIC QUANTITY

By Burton McCollum

In the absolute measurement of current two classes of methods are available which are capable of considerable accuracy, viz, electrodynamometer methods, in which the torque exerted between a fixed and a movable coil is measured and the current in the coils calculated from this measured torque and the known dimensions of the coil; and current balances, in which the force exerted between two coils carrying the current to be measured is balanced against a known weight and the current calculated from the weight and previously determined constants of the coils. By the latter form of apparatus measurements of current have been made in which a higher degree of accuracy has been attained than with the first-named type. There are, however, numerous possible sources of error in the use of the current balance, and, although most of these can be made small individually, they may on account of their number, introduce uncertainty in the result.

The ordinary electrodynamometer methods as well as the current balance require in effect two separate experiments, the first for the determination of the value of the current in terms of a torque or force, and the second for the recording of this current in terms of the electro-chemical equivalent of silver or of a resistance and standard cell. This second experiment may introduce additional possibilities of error, owing to variations in the strength of the current or the voltage of the standard cell, errors in the resistances of the potentiometer circuits and thermo-electromotive forces in the galvanometer and potentiometer circuits, etc., and while these errors can be made small, still their possible presence increases the difficulties of the measurements.

Although the current balance is capable of yielding a high degree of accuracy, it is not desirable to depend on the value given by a single type of apparatus, even though the experiments may be repeated with good agreement by different experimenters and with different instruments. It is highly desirable that if possible the values be checked by some other method, using a totally different type of apparatus.

In electrodynamometers of the ordinary types the dimensions of the windings must be known with extreme accuracy, thus requiring that the coils not only be wound with great regularity and measured with great care, but permanency of both size and form must be assured. In the Gray electrodynamometer a greater difficulty still is met with in measuring the torque. This is accomplished by balancing the torque due to the current against the torsion of a wire, the constant of which has been determined by a separate experiment. The great difficulty in this lies in the erratic behavior of all materials available for the torsion wire. They can not be depended upon to remain constant with time, they vary considerably even with slight changes in temperature, and they all have a greater or less tendency to take a permanent set when twisted, as they must be, in use. Again, both electrodynamometers and current balances have a rather troublesome temperature coefficient, and, although artificial cooling may alleviate this condition somewhat, a certain amount of local heating is inevitable, and the consequent change in the dimensions may give rise to considerable variation in the constants of the instruments between the time of measuring the dimensions of the coils and the time of making a measurement of current.

In what follows, there is described a type of electrodynamometer which, when used to measure the electrochemical equivalent of silver, appears to be free from the objections referred to above, and at the same time it is believed that no other difficulties of a serious nature are introduced.

The instrument consists essentially of a relatively large fixed coil, C_1 Fig. 1, with its axis in a horizontal position, at the center of which is suspended the movable coil C_2, with its axis parallel to the axis of the fixed coil when at rest. Attached to the movable coil is a cylinder K, of some homogeneous material placed with its axis vertical and coincident with the axis of suspension. This cylinder should be of very regular dimensions, and should have a moment of inertia as large as, or larger than, that of the movable coil. If these two coils be connected in series in proper relation, and a current sent through them, there will exist a couple tending to hold the movable coil with its axis parallel to that of the fixed coil, and if the movable coil be given an angular displacement and released, it will oscillate as a torsion pendulum. If now, the form of the field due to the fixed coils be such that the restoring couple is directly proportional to the angular displacement, a condition

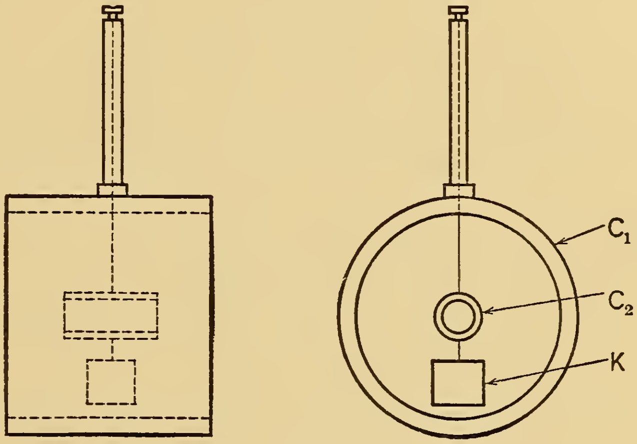

Fig. 1

that can readily be realized as explained later, the period of the oscillation will be independent of the amplitude, and we will have, in effect, a torsion pendulum subject to the laws of a damped oscillation.

The differential equation applying to this system is the well-known form

$$\frac{d^2\theta}{dt^2}+\frac{a}{K}\frac{d\theta}{dt}+\frac{R}{K}\theta=0$$

where R is the restoring couple at unit angular displacement, a is the counter torque due to the damping forces at unit angular velocity and K is the moment of inertia of the moving system. From the well-known integral of this equation, it is readily derived that the time of vibration of the moving system is given by the equation

$$T=\frac{2\pi}{\sqrt{\frac{R}{K}-\frac{a^2}{4K^2}}} \qquad (1)$$

Since, as assumed above, the coils are so proportioned that the torque due to the current i is proportional to the deflection θ, we will have

$$R = Ci^2 + b$$

where C is a constant of the coils, b the constant of torsion of the suspending wire, and R the total restoring couple. Putting this value of R in equation (1), we have,

$$T_1 = \frac{2\pi}{\sqrt{\frac{Ci^2 + b}{K} - \frac{a^2}{4K^2}}} \tag{2}$$

If, now, the current be switched off and the time of vibration again taken, we will have, since i in equation (2) becomes zero,

$$T_2 = \frac{2\pi}{\sqrt{\frac{b}{K} - \frac{a^2}{4K^2}}} \tag{3}$$

Eliminating $\left(\frac{b}{K} - \frac{a^2}{4K^2}\right)$ between (2) and (3), and solving for i, we get

$$i = \frac{2\pi}{T_1}\sqrt{\frac{K}{C}\left(1 - \frac{T_1^2}{T_2^2}\right)} \tag{4}$$

Equation (4) is an exact one, and it shows that damping introduces no error in the calculated value of the current.

If the apparatus be properly designed, the ratio $\frac{T_1^2}{T_2^2}$ can be made small compared to unity (not exceeding 1 per cent), so that a very large error in determining this ratio would introduce but a very small error in the value of i; and, further, since the ratio can be determined with a high degree of accuracy, the factor in parentheses need not be regarded as an appreciable source of error. The ultimate accuracy with which i can be measured depends, therefore, on the accuracy with which the three quantities T_1, K, and C can be measured. These quantities can be determined with a high degree of precision by methods to be indicated later; and, further, since K and C appear under the radical, any error appearing in these values is reduced one half in the value of i. This apparatus and equation (4) may be used for the absolute measurement of current, if such is desired. However, since it is impossible to record the current directly, its value is usually determined only in order that it may be reproduced and maintained constant for a sufficient length of time to enable the electro-chemical equivalent of silver to be determined. This additional experiment is objectionable mainly because it introduces several possible sources of error, as pointed out above; but it is necessary with all forms of apparatus that have heretofore been used for the absolute determination of the electrochemical equivalent of silver. It is not necessary, however, with the type of electro-dynamometer here considered.

The time of vibration T_1, which appears in the denominator of equation (4) may be written $\frac{t}{n}$, where t is the total elapsed time and n is the number of vibrations occurring during that time. Equation (4) then becomes

$$i = \frac{2\pi n}{t}\sqrt{\frac{K}{C}\left(1 - \frac{T_1^2}{T_2^2}\right)}$$

$$\therefore\ it = Q = 2\pi n\sqrt{\frac{K}{C}\left(1 - \frac{T_1^2}{T_2^2}\right)} \qquad (5)$$

where Q is the total quantity of electricity that has passed through the instrument while the movable system is making n vibrations. If, therefore, the instrument be connected in series with a silver voltameter and a current sent through them for any suitable period of time, during which the number of vibrations n is counted, the total quantity of electricity that has passed through the voltameter can be calculated from equation (5). Thus the instrument may be used for the direct measurement of electric quantity. By using equation (5) as the working formula, a very high degree of accuracy should be possible. An inspection of the equation will show that it is not necessary that the current be maintained rigorously constant. The only factor in the equation for Q that varies with i is the factor T_1 under the radical, which varies nearly inversely as the current i. Differentiating equation (5) with respect to the variable T_1, and dividing by Q, we get

$$\frac{dQ}{Q} = -\frac{T_1^2}{T_2^2 - T_1^2} \cdot \frac{dT_1}{T_1} \tag{6}$$

As pointed out above, T_1^2 need not exceed one one-hundredth of T_2^2, and, therefore, it is seen from equation (6) that any given percentage change in T_1 (and therefore in i) would produce in Q a percentage change only about one one-hundredth part as great. Hence, the current might fluctuate continuously within a range of one-fifth per cent, or it might remain continuously too small or too large by as much as one part in a thousand, and the resulting error in Q would not exceed one part in one hundred thousand. It is evident, therefore, that the quantity in parentheses will not be an appreciable source of error, and no great care is necessary in maintaining a definite value of current. The ultimate accuracy with which Q can be determined depends, therefore, on the accuracy with which n, K, and C can be determined. We shall now describe suitable methods for determining these factors.

Determination of n

A typical arrangement of the apparatus is shown in Fig. 2. B is a battery supplying current through the regulating resistance r_1, to the coils C_1 and C_2 of the electro-dynamometer connected in series. S_1 is a single pole double throw switch which at starting is thrown over to the terminal d, so that the current is made to flow through the control coil C_3 which holds the movable coil C_2 at any suitable initial deflection (which need not be measured). The resistance of the circuit df should be previously adjusted nearly equal to that of the circuit ef, which contains the voltameter VA, and the terminals d and e adjusted so that the switch S_1, when thrown over, makes contact with e an instant before breaking the current at d. When ready to start the experiment, the switch S_1 is quickly thrown from d to e which starts the current in the voltameter circuit ef, and at the same time breaks the circuit of the control coil C_3, thus releasing the movable coil C_2 and the vibrations begin. The instrument should then be permitted to swing freely for a sufficient length of time to permit the deposition of enough silver for accurate weighing, during which time the number of complete vibrations of the movable coil is counted. When it is desired to stop the experiment, a reading should be taken of the last maximum deflection θ_0 at the end of n_0 complete vibrations, and about a quarter period later the current should be switched off, and at the same instant the position of the movable coil in its cycle, and also its direction of motion should be noted. From these data, the value of n, the total number of vibrations, can be readily determined as indicated later. In order to determine the position of the coil at the instant of opening the circuit, a telescope and scale g, and reflecting mirror, should be provided as in ordinary galvanometer work. Since the scale is in motion, its position

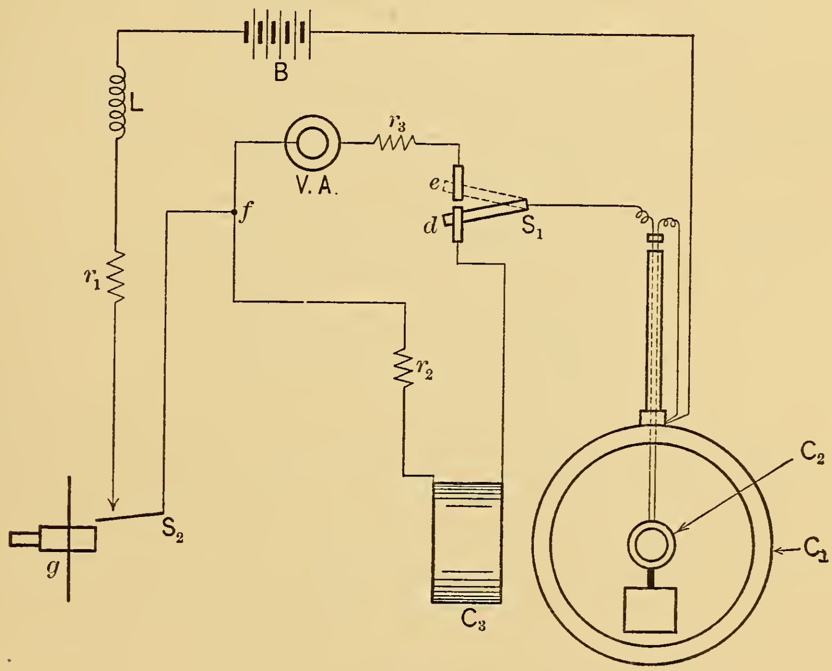

Fig. 2

must be determined by obtaining a momentary view of it at the instant of opening the circuit. Numerous ways suggest themselves for accomplishing this, one of the simplest, perhaps, being as follows: A portion of the scale, preferably the central portion, should be somewhat heavily shaded so that its reflection in the mirror will be very faint, if not invisible. A specially designed switch, S_2, which is used to open the circuit, should be placed near the shaded portion, so that the flash caused by opening the switch will illuminate the scale for a brief instant. If, then, the switch be opened when the shaded portion of the scale is being reflected into the telescope, the scale will appear stationary, and its deflection θ, can be easily read. The brightness of the flash can be made ample by including a self inductance L, in series with the circuit, if necessary.

Assuming, as above, that n_0 is the total number of complete vibrations, T_1 the time of a complete vibration, and t the time that elapses between the completion of n_0 complete periods and the opening of the circuit, we will have, obviously,

$$n = n_0 + \frac{t}{T_1} \tag{7}$$

where n is the factor to be used in equation (6). The ratio $\frac{t}{T_1}$ can be most accurately determined from the last maximum deflection θ_0, and the final deflection θ, taken as described above. From the well-known laws of the torsion pendulum, we have,

$$\theta = \theta_0 \cos \sqrt{\frac{R}{K} - \frac{a^2}{4K^2}} \cdot t$$

$$\therefore \quad \sqrt{\frac{R}{K} - \frac{a^2}{4K^2}} \cdot t = \cos^{-1}\frac{\theta}{\theta_0}.$$

But,

$$\sqrt{\frac{R}{K} - \frac{a^2}{4K^2}} = \frac{2\pi}{T_1}$$

$$\therefore \quad \frac{t}{T_1} = \frac{1}{2\pi} \cos^{-1}\frac{\theta}{\theta_0}$$

$$\therefore \quad n = n_0 + \frac{1}{2\pi} \cos^{-1}\frac{\theta}{\theta_0} \tag{8}$$

It is important to determine the probable accuracy with which n can be determined, and the conditions under which the greatest precision may be attained. Differentiating equation (8), we have,

$$dn = -\frac{1}{2\pi\sqrt{1 - \left(\frac{\theta}{\theta_0}\right)^2}} \cdot d\left(\frac{\theta}{\theta_0}\right)$$

$$\therefore \quad \frac{dn}{n} = -\frac{1}{2\pi\sqrt{1 - \left(\frac{\theta}{\theta_0}\right)^2}\left(n_0 + \frac{1}{2\pi} \cos^{-1}\frac{\theta}{\theta_0}\right)} \cdot d\left(\frac{\theta}{\theta_0}\right) \tag{9}$$

From this it appears that the greatest accuracy is attainable if the final reading is taken when θ is small, that is, when the coil is passing near the middle portion of its swing. Assuming that the experiment has continued for at least half an hour, and that the period of vibration is, say, fifteen seconds, thus giving a value of n_o at least one hundred twenty, and assuming further that the final reading is taken when θ is between $+\frac{1}{10}\theta_o$ which would not be difficult, we may write equation (9) with close approximation:

$$\frac{dn}{n} = -\frac{1}{2\pi n_o} \cdot d\left(\frac{\theta}{\theta_o}\right)$$

From which we have the two relations:

$$(a), \frac{dn}{n} = -\frac{d\theta}{2\pi n_o \theta_o} = \frac{dx}{4\pi n_o \lambda \theta_o}. \qquad (10)$$

where x is the length of the arc θ and λ the distance from scale to mirror; and,

$$(b), \frac{dn}{n} = \frac{1}{2\pi n_o} \cdot \frac{\theta}{\theta_o} \cdot \frac{d\theta_o}{\theta_o} \qquad (11)$$

For the purpose of making an approximate numerical calculation of the probable error in measuring n, we may assume that the scale is two meters from the mirror and that θ_0 equals $\frac{1}{25}$ radian. Putting these values in equation (10), we have

$$\frac{dn}{n} = \frac{\pm 25}{4\pi . 120 . 200} . dx = \frac{\pm 1}{12057} . dx$$

It follows, therefore, that an error as great as one millimeter in the final reading of the scale would introduce an error of less than one part in 120,000 in the value of n.

Substituting the assumed values in equation (11), we get

$$\frac{dn}{n} = \pm \frac{1}{7536} \cdot \frac{d\theta_0}{\theta_0}$$

Hence, any small error in reading θ_0 is reduced over 7500 times in the value of n. With reasonable care, therefore, the experimental error in determining n would be altogether negligible, and the precision of the measurement of Q would depend only on the determination of K and C.

Determination of K

In order that K may be determined accurately, the movable coil should have attached to it a cylinder of some homogeneous material suspended with its axis vertical and coincident with the line of suspension. It should be made of some material that is known to be of uniform density and should preferably be of high electrical resistance so as to avoid excessive damping. Brass and other metal castings are objectionable because of the possibility of blow holes and other defects which can not be detected. Forged metal would be better, but there would still be uncertainty in regard to the uniformity of the material, and, further, the damping might become troublesome, for, although damping is not directly a source of error, it might, if excessive, bring the system to rest before enough time has elapsed to permit the deposition of a sufficient quantity of silver. Glass would seem to be a suitable material for this purpose. A carefully made cylinder of glass would be expensive, but it would have the great advantage that any inequalities could readily be detected by simple optical tests. It would also be free from damping effects due to induced currents. The moment of inertia of such a cylinder carefully ground to size could be determined with great precision by calculation from its mass and dimensions. The most obvious method of determining the total moment of inertia of the system would be to take readings of the time of vibration, first without the glass cylinder attached and again with this cylinder in place, and since the moments of inertia are proportional to the squares of the times of vibration, we would have at once, if K_1 and K_2 are the moments of inertia of the coil and cylinder respectively,

$$\frac{K_1}{K_2+K_1}=\left(\frac{T_1}{T_2}\right)^2 \tag{12}$$

There are two objections to this method, however. The chief objection is that the torsion wire is subjected to considerably greater tension in the second part of the experiment than in the first, and this would undoubtedly affect appreciably the torsion of the wire. This disturbance could be greatly reduced by using a multifilar suspension, but even then it would probably be troublesome. Further, it is practically impossible to find a torsion wire that can be depended on with certainty to remain constant even under constant stress, so that this method would give rise to considerable uncertainty, even though precautions were taken to keep the tension of the wire constant. These errors can be almost entirely eliminated, however, by making a constant current flow through the coils while taking the period. There would be no difficulty in making the electro-magnetic couple at least 99 per cent of the total couple, and, therefore, any change in the constant of torsion of the wire would be reduced one hundred times in its effect on the value of the ratio of the moments of inertia as calculated from equation (12). Equation (12) then holds, and the total moment of inertia $K = K_1 + K_2$, can be readily calculated if K_2 has been determined. It should be noted that the value of K thus determined is independent of any absolute errors in the instrument used for measuring time, provided that the instrument remains consistent with itself. Further, the moment of inertia of the air within and near the moving coil is taken into account in this way.

Determination of C

The only other factor that remains to be determined is C, the constant of the coils. In order to determine C, we shall make use of the formula given by Gray [1] for the mutual inductance between two coils, viz:

$$M = \pi^2 n_1 n_2 a^2{}_1 a^2{}_2 (K_1 k_1 Z_1 + K_2 k_2 Z_2 + K_3 k_3 Z_3 + \ldots\ldots) \qquad (13)$$

As pointed out by Gray, if the coils are concentric, the even terms all vanish and (13) becomes

$$M = \pi^2 n_1 n_2 a^2{}_1 a^2{}_2 (K_1 k_1 Z_1 + K_3 k_3 Z_3 + K_5 k_5 Z_5 + \ldots\ldots) \qquad (14)$$

[1] Gray, Absolute Measurements in Electricity and Magnetism, Vol. 2, part 1, pp 274, 275.

$$\left.\begin{array}{l}
\left.\begin{array}{l}\text{where } K_1 = \dfrac{4\rho_1}{a_1 D} \\ k_1 = 2\rho_2 a_2\end{array}\right\}(a) \\
\left.\begin{array}{l}K_3 = -\dfrac{\rho_1 a_1}{D^5} \\ k_3 = \rho_2 a^3{}_2(4\rho^2{}_2 - 3)\end{array}\right\}(b) \\
K_5 = -\dfrac{\rho_1 a^3{}_1}{4D^9}(4\rho^2{}_1 - 3) \\
k_5 = 2\rho_2 a^5{}_2\left(2\rho^4{}_2 - 5\rho^2{}_2 + \dfrac{5}{4}\right) \\
K_7 = -\dfrac{\rho_1 a^5{}_1}{4D^{13}}\left(4\rho^4{}_1 - 10\rho^2{}_1 + \dfrac{5}{2}\right) \\
k_7 = \rho_2 a^7{}_2\left(4\rho^6{}_2 - 21\rho^4{}_2 + \dfrac{35}{2}\rho^2{}_2 - \dfrac{35}{16}\right)
\end{array}\right\} \qquad (15)$$

where ρ_1 and ρ_2 are the ratios of length to diameter, n_1 and n_2 the number of turns per centimeter, and a_1 and a_2 the radii of the fixed and movable coils respectively, and D the half diagonal of the fixed coil. Z_1, Z_2, Z_3, etc., are the zonal surface harmonics corresponding to the respective terms, with the angle between the axes of the coils as argument.

Differentiating equation (14) with respect to θ, we get, as the equation for torque,

$$\begin{aligned} T &= i^2\frac{dM}{d\theta} \\ &= \pi^2 n_1 n_2 a^2{}_1 a^2{}_2 i^2\left(K_1 k_1 \frac{dZ_1}{d\theta} + K_3 k_3 \frac{dZ_3}{d\theta} + K_5 k_5 \frac{dZ_5}{d\theta} + \ldots\right) \end{aligned} \qquad (16)$$

Equating the expressions for K_5 and k_7 from equations (15) to zero and solving for ρ_1 and ρ_2, we get,

$$\left.\begin{array}{l} \rho_1 = \dfrac{\sqrt{3}}{2} \\ \rho_2 = \begin{cases} 2.062 \\ .92 \\ .38 \end{cases} \end{array}\right\} \qquad (17)$$

If then the coils are so wound that the ratio of length to diameter of the fixed coil is $\frac{\sqrt{3}}{2}$ and that of the movable coil either of the values of ρ_2 given in equation (17), the third and fourth terms in equation (16) disappear, and if the movable coil is small compared to the fixed coil, all the terms beyond the fourth are altogether negligible. Equation (16) therefore becomes,

$$T=\pi^2 n_1 n_2 a^2{}_1 a^2{}_2 i^2\left(K_1 k_1 \frac{dZ_1}{d\theta}+K_3 k_3 \frac{dZ_3}{d\theta}\right) \tag{18}$$

$$=\frac{2\pi^2 N_1 N_2 a^2{}_2 i^2}{D}\left(\frac{dZ_1}{d\theta}+R_3 \frac{dZ_3}{d\theta}\right) \tag{19}$$

where N_1 and N_2 are the total number of turns on the two coils, and R_3 is, from equations (15), (*a*) and (*b*), equal to—

$$R_3=\frac{K_3 k_3}{K_1 k_1}=-\frac{a^2{}_2(4\rho^2{}_2-3)}{8a^2{}_1(1+\rho^2{}_1)^2} \tag{20}$$

In the beginning it was assumed that $T=-Ci^2\theta$, and if this condition is to be satisfied, we must have from equation (19),

$$-C\theta=\frac{2\pi^2 N_1 N_2 a^2{}_2}{D}\left(\frac{dZ_1}{d\theta}+R_3 \frac{dZ_3}{d\theta}\right) \tag{21}$$

It is not necessary that this equation shall hold for all values of θ, but only throughout the range of deflection used in actual experiment. An angle of oscillation of 6 or 8 degrees is entirely ample for present purposes, so that if equation (21) holds for all values of θ up to ± 4 degrees, it will be sufficient. We shall now determine the value of R_3 that will make equation (21) hold within these, and even greater limits.

Since $Z_1=\cos\theta$ and $Z_3=\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta$,

$$\therefore \frac{dZ_1}{d\theta}=-\sin\theta$$

and $\dfrac{dZ_3}{d\theta}=-\dfrac{15}{2}\cos^2\theta\sin\theta+\dfrac{3}{2}\sin\theta.$

Expressing these in the form of series, we have,

$$\frac{dZ_1}{d\theta}=-\sin\theta=-\theta\left(1-\frac{\theta^2}{6}+\frac{\theta^4}{120}-\frac{\theta^6}{5040}+\ .\ .\ .\ .\ .\right)$$

$$\frac{dZ_3}{d\theta}=-\theta\left(6-\frac{17}{2}\theta^2+\frac{19}{5}\theta^4-\frac{1367}{1680}\theta^6+\ .\ .\ .\ .\right)$$

Substituting these values in equation (21) above, we have,

$$C\theta=\frac{2\pi^2N_1N_2a^2{}_2\theta}{D}\Big[1-\frac{\theta^2}{6}+\frac{\theta^4}{120}-\frac{\theta^6}{5040}+\ \ldots$$

$$+R_3\Big(6-\frac{17}{2}\theta^2+\frac{19}{5}\theta^4-\frac{1367}{1680}\theta^6+\Big)\Big]$$

$$=\frac{2\pi^2N_1N_2a^2{}_2\theta}{D}\Big[1+6R_3-\theta^2\Big(\frac{1}{6}+R_3\frac{17}{2}\Big)+\theta^4\Big(\frac{1}{120}+R_3\frac{19}{5}\Big)-\ldots\Big] \qquad (22)$$

If now we assign such a value to R_3 that the terms in θ^2 vanish, we will have,

$$R_3\frac{17}{2}+\frac{1}{6}=0.$$

$$\therefore\ R_3=-\frac{1}{51} \qquad (23)$$

Putting this value in (22) and reducing, we get,

$$C=\frac{2\pi^2N_1N_2a^2{}_2}{D}\Big(\frac{45}{51}-\frac{9}{136}\theta^4+\frac{15}{952}\theta^6-\ldots.\Big) \qquad (24)$$

For all values of θ between ± 5 degrees, the sum of all the terms involving θ in equation (24) amounts to less than .0000038, or about 4.3 parts in a million in comparison with the constant term. Even up to deflections of ± 8 degrees the time of vibration is independent of amplitude to within about one part in one hundred thousand. Hence, all these terms may be neglected even for the most accurate work, and the assumption of a constant value of C is therefore justified. We then have,

$$C=\frac{2\pi^2N_1N_2a^2{}_2}{D}\cdot\frac{45}{51} \qquad (25)$$

We are now in a position to determine which of the three values of ρ_2 given in equation (17) may be used. Whichever value is used must satisfy equation (17), and be consistent with equations (20) and (23). Substituting the value of R_3 from equation (23) in equation (20), and making $\rho_1=\frac{1}{2}\sqrt{3}$ according to equation (17), and finally inserting in turn the three values of ρ_2 and solving for $\frac{a_2}{a_1}$, we get, approximately,

$$\frac{a_2}{a_1}=\frac{1}{\sqrt{-5.04}}\ldots(\rho_2=.38)$$

$$\frac{a_2}{a_1}=1.1\ldots\ldots(\rho_2=.92)$$

$$\frac{a_2}{a_1}=\frac{1}{5.4}\ldots.(\rho_2=2.062).$$

It is obvious at once that the first two of these values can not be used, since the first is imaginary, and the second gives a value for the radius of the movable coil 10 per cent greater than that of the fixed coil, which is obviously impracticable. The third value requires that the radius of the fixed coil be 5.4 times that of the movable coil, a value very desirable in practice, and one which not only insures that the neglect of the higher terms will not introduce any appreciable error, but it also gives such small values to the coefficients of the third and fourth terms of equation (16) that a comparatively large error in proportioning the coils would not appreciably affect the vanishing of these terms, as will presently be shown.

The value of C may be calculated from equation (25) if desired, but to do so requires a highly accurate knowledge of the dimensions of the coils. A better way of determining C is as follows:

Referring to equation (14), and noting that only the first two terms have any appreciable value under the conditions imposed, we have,

$$\mathrm{M}=\pi^2 n_1 n_2 a^2{}_1 a^2{}_2(K_1k_1Z_1+K_3k_3Z_3)$$

$$=\frac{2\pi^2N_1N_2a^2{}_2}{D}\left(Z_1+\frac{K_3k_3}{K_1k_1}Z_3\right)=\frac{2\pi^2N_1N_2a^2{}_2}{D}(Z_1+R_3Z_3)$$

If the coil be placed in the position of maximum mutual inductance M_o, we will have, since $\theta=0$ and hence $Z_1=Z_3=1$,

$$M_o=\frac{2\pi^2N_1N_2a^2{}_2}{D}(1+R_3)=\frac{2\pi^2N_1N_2a^2{}_2}{D}\cdot\frac{50}{51} \tag{26}$$

Dividing (25) by (26), and solving for C, we get,

$$C=\frac{9}{10}M_o \tag{27}$$

It is necessary, therefore, to measure only the maximum mutual inductance of the coils for the determination of the constant of the instrument by equation (27). It is thus possible not only to determine the constant of the instrument by simply measuring the maximum mutual inductance, but there is the additional important advantage that the constant can be quickly redetermined electrically at any time without disturbing the coils. Substituting the value of C derived from equation (27) in equation (5), we have, as the final working formula,

$$Q = 2\pi n\sqrt{\frac{10K}{9M_0}\left(1-\frac{T_1^2}{T_2^2}\right)} \tag{28}$$

DISCUSSION OF SOURCES OF ERROR

TEMPERATURE EFFECTS

It is evident that if we maintain the fixed coil at a constant temperature, as by water cooling or otherwise, the field at the center will be constant and the mutual inductance will vary nearly as the square of the linear dimensions of the movable coil. But the value of K is also proportional to the square of the linear dimensions of the moving system. The ratio of K to M_0 (28) would, therefore, be very nearly independent of temperature, and since these are the only quantities in the equation that are affected by temperature, the temperature coefficient of the instrument would be extremely small if not entirely negligible. Further, the mutual inductance can be readily measured immediately before and again immediately after the experiment, so that any slight drift in the constant of the instrument can be detected.

DIMENSIONS OF THE COILS

Since the above equation for C is derived on the assumption of certain relations between the dimensions of the coils, it is important to note what effect slight errors in these dimensions may have on the calculated value of electric current or quantity.

The neglected even terms need not cause any trouble, since each of these terms contains two factors, both of which approach zero as the centers of the coils are made to approach the intersection of the axes. This adjustment being entirely independent of the dimensions of the coils can, with a little care, be made as close as desired, and there would be no difficulty in making these terms altogether negligible.

Following the method given by Gray,[2] we have for the general integrals for the ninth term of Gray's series for torque,

$$\frac{1}{n(n+1)}\int\frac{\psi Z'_n dx}{D^{n+2}}=-\frac{\rho_1 a^7_1}{11520\,D^{17}}(576\rho^6_1-3024\rho^4_1+2520\rho^2_1-315)$$

$$\int D^{n}{}_2{}^{-1}\psi' Z'_n dx=\frac{15\rho_2 a^9_2}{128}\left(\frac{128}{3}\rho^8_2-384\rho^6_2+672\rho^4_2-280\rho^2_2+21\right)$$

Substituting the limits, and substituting $\rho_1=\frac{\sqrt{3}}{2}$ and $\rho_2=2.062$, and dividing by K_1k_1, we have,

$$\frac{K_9k_9}{K_1k_1}=\frac{1}{5,313,000}. \tag{29}$$

By Roderigues Theorem, we have

$$Z_9=\frac{1}{2^9.9!}\cdot\frac{d^9}{dZ^9_1}(Z^2_1-1)^9$$

where, as above, $Z_1=\cos\theta$. Hence, we have,

$$\frac{dZ_9}{dZ_1}=\frac{1}{2^9.9!}\cdot\frac{d^{10}}{dZ_1{}^{10}}(Z^2_1-1)^9$$

Carrying out this differentiation, and substituting as close approximation, $\cos\theta=1$, and putting the value of $\frac{dZ_9}{dZ_1}$ thus obtained in (16) together with the value of $\frac{K_9k_9}{K_1k_1}$ from (29), we find that the ninth term amounts to a little less than 1 part in 120,000 in comparison with the first term.

We shall now investigate the error that may arise due to neglecting the third and fourth terms of equation (16). The ratio of the third term to the first term is

$$\mu_5=\frac{K_5k_5}{K_1k_1}\cdot\frac{\frac{dZ_5}{d\theta}}{\frac{dZ_1}{d\theta}}$$

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[2] Gray, Absolute Measurements in Electricity and Magnetism, Vol. **2**, part 1, pp. 273, 274.

Therefore $\mu_5 = -\frac{a^4_2}{32a^4_1}\left(4\rho^4_2 - 10\rho^2_2 + \frac{5}{2}\right)\cdot\frac{dZ_5}{dZ_1}\cdot\frac{4\rho^2_1 - 3}{(1+\rho^2_1)^4}$ (30)

and since $Z_5 = \frac{63}{8}\cos^5\theta - \frac{35}{4}\cos^3\theta + \frac{15}{8}\cos\theta$

and $Z_1 = \cos\theta$

$$\therefore \frac{dZ_5}{dZ_1} = \frac{315}{8}\cos^4\theta - \frac{105}{4}\cos^2\theta + \frac{15}{8}$$

Since θ is always very small, $\cos\theta$ will differ but slightly from unity, and we have, without appreciable error,

$$\frac{dZ_5}{dZ_1} = 15$$

Substituting this value of $\frac{dZ_5}{dZ_1}$ in equation (30) together with the numerical values of the other factors as given above, viz: $\rho_1 = \frac{\sqrt{3}}{2}$, $\rho_2 = 2.062$, $\frac{a_1}{a_2} = 5.4$, we have, with close approximation,

$$\mu_5 = -\frac{1}{527}(4\rho^2_1 - 3) \tag{31}$$

In any case in which an accurate measurement is sought, the length of the fixed coil would be of the order of perhaps 50 centimeters, with the diameter greater in the ratio of $\frac{2}{\sqrt{3}}$. Choosing these as a basis for numerical calculation, we find that if the actual length of the coil differs from its assumed value by as much as half a millimeter, which would give an error of one part in one thousand in the ratio of ρ_1, the value of μ_5 from equation (31) would amount to only 1.1 parts in 100,000 in comparison with the first term. Since this is an error in the torque, it appears only half as great in the value of current or electric quantity, and hence is entirely negligible. Obviously, a somewhat smaller error would result if an error of one-half millimeter should occur in the diameter of the coil.

For the seventh term we have

$$\mu_7 = \frac{K_7 k_7}{K_1 k_1} \cdot \frac{\frac{dZ_7}{d\theta}}{\frac{dZ_1}{d\theta}}$$

$$= -\frac{a^6_2}{32 a^6_1} \cdot \frac{4\rho^4_1 - 10\rho^2_1 + 5/2}{(1+\rho^2_1)^6} \cdot \frac{dZ_7}{dZ_1} \cdot \left(4\rho^6_2 - 21\rho^4_2 + \frac{35}{2}\rho^2_2 - \frac{35}{16}\right) \quad (32)$$

and since $Z_7 = \frac{429}{16}\cos^7\theta - \frac{693}{16}\cos^5\theta + \frac{315}{16}\cos^3\theta - \frac{35}{16}\cos\theta$

$$\therefore \frac{dZ_7}{dZ_1} = \frac{3003}{16}\cos^6\theta - \frac{3465}{16}\cos^4\theta + \frac{945}{16}\cos^2\theta - \frac{35}{16}$$

Hence, for small values of θ, we have approximately

$$\frac{dZ_7}{dZ_1} = 28$$

Substituting this, and the proper numerical values of a_1, a_2, and ρ_1 in equation (32), we get

$$\mu_7 = +\frac{77 \cdot 10^{-5}}{216}\left(4\rho^6_2 - 21\rho^4_2 + \frac{35}{2}\rho^2_2 - \frac{35}{16}\right) \quad (33)$$

On the basis of the preceding assumption of 50 centimeters as the length of the fixed coil, the movable coil would have a length of a little less than 20 centimeters. An error of 1 millimeter here would be about one-half per cent, and assuming that ρ_2 is one-half per cent greater or less than its proper value of 2.062, we would have from equation (33), with substantial accuracy,

$$\mu_7 = 9 \times 10^{-6}$$

An error of 1 millimeter in the length of the movable coil would therefore give the seventh term a value equal to, roughly, 1 part in 110,000 in comparison with the first term, with a resulting error of 1 part in 220,000 in the quantity being measured. The same percentage error in the radius of the movable coil would give rise to an equal error.

It appears, therefore, that if the coils are wound and mounted with reasonable care, no correction will be necessary for the neglected terms, even in work of the highest accuracy, and the simple form of equation (18) and those derived from it, may be regarded as substantially exact. It remains, therefore, to determine only the errors that may arise in the use of this equation.

From equation (22), we have, after omitting the higher terms,

$$C\theta = \frac{2\pi^2 N_1 N_2 a^2{}_2 \theta}{D}(1 + 6R_3)$$

$$\therefore C = \frac{2\pi^2 N_1 N_2 a^2{}_2}{D}(1 + 6R_3) \tag{34}$$

and from equation (14), we get

$$M_o = \frac{2\pi^2 N_1 N_2 a^2{}_2}{D}(1 + R_3) \tag{35}$$

Hence from (34), $C = \dfrac{M_o(1 + 6R_3)}{1 + R_3}$

and, therefore, from (5)

$$Q = 2\pi n\sqrt{\frac{K(1 + R_3)}{M_o(1 + 6R_3)}\left(1 - \frac{T^2{}_1}{T^2{}_2}\right)} \tag{36}$$

Differentiating this equation with respect to R_3, and dividing the resulting equation by (36), member for member, we have,

$$\frac{dQ}{Q} = -\frac{5}{2}\cdot\frac{dR_3}{1 + 7R_3 + 6R^2{}_3} = -2.9dR_3 \tag{37}$$

From equation (20), we have,

$$R_3 = -\frac{a_2^2}{8a_1^2}\cdot\frac{4\rho_2^2 - 3}{(1 + \rho_1^2)^2}$$

Differentiating this equation with respect to the four quantities ρ_2, ρ_1, a_2, and a_1, separately, we have,

$$dR_3 = -\frac{a_2^2\rho_2^2}{a_1^2(1 + \rho_1^2)^2}\cdot\frac{d\rho_2}{\rho_2} = -\frac{1}{20.7}\cdot\frac{d\rho_2}{\rho_2}$$

$$dR_3 = +\frac{a_2^2\rho_1^2(4\rho_2^2 - 3)}{2a_1^2(1 + \rho_1^2)^3}\cdot\frac{d\rho_1}{\rho_1} = +\frac{1}{29.6}\cdot\frac{d\rho_1}{\rho_1}$$

$$dR_3 = -\frac{a_2^2(4\rho_2^2-3)}{4a_1^2(1+\rho_2^2)^2}\cdot\frac{da_2}{a_2} = -\frac{1}{25.3}\cdot\frac{da_2}{a_2}$$

$$dR_3 = \frac{a_2^2(4\rho_2^2-3)}{4a_1^2(1+\rho_1^2)^2}\cdot\frac{da_1}{a_1} = \frac{1}{25.3}\cdot\frac{da_1}{a_1}$$

Putting these values in equation (37) successively we have, very closely,

$$\left.\begin{aligned} \frac{dQ}{Q} &= \frac{1}{7.2}\cdot\frac{d\rho_2}{\rho_2} \\ \frac{dQ}{Q} &= -\frac{1}{10}\cdot\frac{d\rho_1}{\rho_1} \\ \frac{dQ}{Q} &= \frac{1}{9}\cdot\frac{da_2}{a_2} \\ \frac{dQ}{Q} &= -\frac{1}{9}\cdot\frac{da_1}{a_1} \end{aligned}\right| \qquad (38)$$

These equations show that any small percentage errors in the value of ρ_1 will be reduced ten times in the calculated value of electric quantity. Errors in ρ_2 are reduced 7.2 times and those in a_2 and a_1 nine times. The same ratios hold in the equation for current. It appears, therefore, that if ordinary care be exercised in measuring the coils, any errors that may arise in these measurements will not affect appreciably the calculated value of electric current or quantity.

It may be well to state briefly the considerations that lead to the adoption of the particular proportions of the coils given above. In the Gray electro-dynamometer it is usual to give both fixed and movable coils a ratio of length to diameter equal to $\frac{\sqrt{3}}{2}$, in which case the third and fifth terms vanish, leaving only the seventh and higher terms, which are made negligibly small by making the movable coil small compared to the fixed coil. This is impracticable here, because, in that case, the torque would be proportional to the sine of the angle of deflection and the period would not be isochronous except for extremely small values of θ. The initial deflection could not exceed about one-half degree, and with the inevitable damping which would occur during a half-hour run the final deflection would be too small to permit of a sufficiently accurate measurement of the value of n. To overcome this, one term after the first must be used in order to make the period independent of the deflection. By using the third term as described above, we can use deflections about sixteen times greater than would be possible with the Gray type of instrument before the period becomes appreciably affected by the amplitude of the vibration. If we desired to do so, we could choose such values of ρ_1 and ρ_2 that would cause the third and seventh terms to disappear and use the fifth term to produce the condition of isochronous vibration. This case has been worked out, but is found to be less suitable than the one given above for two reasons. In the first place, the limiting value of the deflection that can be used is only about two-thirds as great. The most serious objection, however, lies in the fact that on working out the approximation formulæ corresponding to equation (38), we find that the dimensions of the movable coil have to be determined with an accuracy about three times as great as in the present case in order to secure the same ultimate accuracy in the measurement of Q.

A possible method would be to make the third and fifth terms disappear as in the Gray electro-dynamometer and use the seventh term for making the time of vibration independent of the amplitude. To do this, however, would require that the radius of the movable coil be so large that the ninth and eleventh terms become troublesome. It seems, therefore, that the method described above of reserving the third term and making the others vanish possesses such important advantages as to make it the only one worthy of serious consideration in any practical case.

EFFECT OF EARTH'S MAGNETIC FIELD

To eliminate the effect of the earth's magnetic field, it is necessary only to permit the normal working current to flow through the movable coil alone, while the value of T_2 is being determined. In this way the effect of the earth's field is eliminated along with the torsion of the suspending wire. It is true that in this case the time of vibration is not strictly isochronous, since the component of torque due to the earth's field varies as the sine of the angle of deflection. But since the angle of vibration is small, the total effect on the time of vibration is extremely small, and since whatever effect there is enters only into the ratio $\frac{T_1}{T_2}$ in equation (5), the error would practically vanish. A simple calculation shows that it could not be greater than a few parts in a million. Likewise, any small variations in the value of the earth's field such as would ordinarily occur would disappear for the same reason. It will not be necessary, therefore, to use a compensating coil to neutralize the earth's field.

ELECTROSTATIC FORCES

Owing to the smallness of the movable coil in comparison with the fixed coil, and to the relatively low voltages used, it is evident that the total electrostatic forces that could act on the movable coil would be very small. Moreover, it is readily seen from the symmetry of the system that when the coils are coaxial any electrostatic force that might exist would be one of translation only, and could not give rise to any couple whatever; and since, in normal working, the angle between the axes of the coils would never be more than 4 or 5 degrees, no disturbance of appreciable magnitude is to be expected here. This point can be tested experimentally, however, by opening the movable coil at the center and applying between the two halves the normal potential drop across the terminals of the movable coil, at the same time permitting the normal working current to flow through the fixed coil. It is readily seen that the electrostatic couple thus set up would be slightly larger than that which would exist under normal working conditions; and if the effect of this couple is found to have an appreciable effect on the period of vibration, the reading thus obtained will permit of a fairly accurate calculation of the true effect. It should be noted that this reading is taken under the most favorable conditions, since, owing to the absence of the electromagnetic couple, the electrostatic couple would be relatively about a hundred times greater in comparison with the total couple than it would be when the current is flowing through both coils. It would thus be very easy to detect an effect that would be entirely imperceptible under normal working conditions.

The foregoing method for the absolute measurement of electric quantity has not yet been subjected to experimental test, so that the limits of the ultimate accuracy attainable by it can not be stated with great precision. Inasmuch, however, as most of the important phases of the problem are amenable to exact mathematical calculation, the chief features of the performance of the instrument can be predicted with considerable confidence.

As compared with the Gray type of electrodynamometer and some types of the current balance, this instrument possesses, in common with the Rayleigh type of current balance, the advantage that errors in the dimensions of the coils are reduced manifold. The Rayleigh balance, however, possesses this advantage in greater degree. The method here described also possesses an advantage over the Gray electrodynamometer in that the torsion of the suspending wires is eliminated. Further, the fact that the indications of the instrument are in a large degree independent of the temperature of the movable coil and of moderate variations in the strength of the working current, contributes greatly to the convenience and to some extent to the accuracy of the method. As to the sensibility of the instrument, it seems that owing to the very great apparent travel of the scale, which, as a rule, would be from 100 to 200 meters, the sensibility of the instrument may safely be said to far exceed the limits of accuracy aimed at in absolute measurements at the present time. The constant of the instrument can be redetermined at any time without disturbing the coils by simply measuring their maximum mutual inductance, and this affords a convenient and accurate check on the stability of the instrument, which would in turn increase considerably the confidence that could be placed in individual measurements. The further fact that the quantity of electricity may be obtained directly without any knowledge of the value of the current being necessary, might often be of advantage where the electrochemical equivalent of silver or other metals is being determined, or, in other cases, where the time integral of current is required.

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